Perturbation Theory for Dual Semigroups V. Variation of Constants Formulas

O. DIEKMANN Centre for Mathematics and Computer Science, Amsterdam, The Netherlands, and Institute for Theoretical Biology, Leiden, The Netherlands

M. GYLLENBERG Institute for Applied Mathematics, Lulea University of Technology, Lulea, Sweden

H. R. THIEME* Department of Mathematics, Arizona State University, Tempe, Arizona

1. INTRODUCTION

If A_0 is the infinitesimal generator of a strongly continuous semigroup $T_0(t), t \geq 0$, on a Banach space X, the dual A_0^* of A_0 is the weak* generator of the dual semigroup $T_0^*(t) = (T_0(t))^*$, $t \geq 0$, on X^* in the following sense:

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$$x^* \in D(A_0^*) \quad \text{and} \quad A_0^* x^* = y^* \tag{1.1}$$

iff

$$\frac{d}{dt}\langle x, T_0^*(t)x^*\rangle = \langle x, T_0^*(t)y^*\rangle\,, \quad x \in X, t \geq 0\,.$$

For any $x^* \in X^*$, $t \geq 0$ we have $\int_0^t T_0^*(r)x^*dr \in D(A_0^*)$ (1.2)

and

$$A_0^*\Big(\int_0^t T_0^*(r)x^*dr\Big) = T_0^*(t)x^* - x^*\,.$$

Here $\int_0^t T_0^*(r)x^*dr$ has to be interpreted as the weak* integral

$$\langle x, \int_0^t T_0^*(r)x^*dr\rangle = \int_0^t \langle T_0(r)x, x^*\rangle dr\,. \tag{1.3}$$

In *Clément et al.* (1989b) it is shown that the perturbed operator $A^\times = A_0^* + C$, where $C : \overline{D(A_0^*)} \to X^*$ is bounded and linear, generates a weakly* continuous semigroup $T^\times$ on X^* in the sense of (1.1) and (1.2). Note that in general $X^\odot := \overline{D(A_0^*)} \neq X^*$. In *Clément et al.* (1989b) the restriction $T^\odot$ of $T^\times$ on $X^\odot$ is constructed first via a variation of constants formula and then extended to the space X^*. In *Clément et al.* (1989c) a general Hille-Yosida type characterization is derived for the weak* generators of weakly* continuous semigroups (in the sense of (1.1) and (1.2)).

In this paper we derive variation of constants formulas for the semigroup $T^\times$ (rather than for its restriction $T^\odot$ to $X^\odot$). The construction presented here — which is independent of the approach in *Clément et al.* (1987, 1989a,b,c) — relies on the observation that $A^\times$ generates an 'integrated semigroup' $S^\times$ on X^* such that $S^\times(t)$ is locally Lipschitz in the operator norm. $S^\times$ can also be described by a variation of constants formula. See *Arendt* (1987), *Kellermann* (thesis), *Kellermann&Hieber* (1989), *Neubrander* (1988), *Thieme* (to appear) for some background material concerning 'integrated semigroups.'

An alternative approach, which does not take the operator C as a starting point but considers 'multiplied integrals' of the dual semigroup T_0^* instead, is presented by *Diekmann, Gyllenberg&Thieme* (preprint).

The formulas derived in this paper will allow easy derivation of certain properties of $T^\times$. We expect that they will play a crucial role in extending the perturbation theory from dual semigroups to dual evolutionary systems and in handling quasilinear Cauchy problems on non-reflexive dual Banach spaces. Such problems arise from physiologically structured population models — see *Metz&Diekmann* (1986) for reference — in which population growth couples back to individual development.

2. BASIC IDEAS AND RESULTS

In *Clément et al.* (1989b) a strongly continuous semigroup $T^\odot$ is constructed on $X^\odot = \overline{D(A_0^*)}$ via the variation of constants formula

$$T^\odot(t)x^\odot = T_0^\odot(t)x^\odot + \int_0^t T_0^*(t-\tau)C\,T^\odot(\tau)x^\odot d\tau, \quad x^\odot \in X^\odot \tag{2.1}$$

with $T_0^\odot$ denoting the restriction of T_0^* to $X^\odot$. Then $T^\odot$ is extended to X^* by the so-called intertwining formula

$$T^\times(t) = (\lambda I - A^\times)T^\odot(t)(\lambda I - A^\times)^{-1}\,.$$

A variation of constants formula of type (2.1) is not possible for $T^\times$ because C is assumed to be defined on $X^\odot$ only. In order to overcome this difficulty we shall justify the following formula in section 6:

$$\begin{aligned} &T^\times(t)x^* \\ &= T_0^*(t)x^* + w^* - \lim_{\lambda\to\infty}\int_0^t T_0^*(t-\tau)C\lambda(\lambda - A_0^*)^{-1}T^\times(\tau)x^*\,d\tau \\ &= T_0^*(t)x^* + w^* - \lim_{\lambda\to\infty}\int_0^t T^\times(t-\tau)C\lambda(\lambda - A_0^*)^{-1}T_0^*(\tau)x^*\,d\tau. \end{aligned} \tag{2.2}$$

$T^\times(t)$ can be represented by a 'generation' expansion

$$T^\times(t) = \sum_{n=0}^{\infty} T_n^\times(t) \tag{2.3}$$

with $T_0^\times = T_0^*$ and

$$T_{n+1}^\times(t)x^* = w^* - \lim_{\lambda\to\infty}\int_0^t T_0^*(t-\tau)C\lambda(\lambda - A_0^*)^{-1}T_n^\times(t)x^*\,d\tau\,. \tag{2.4}$$

The series (2.3) converges in the operator norm. We shall see in section 6 that the $w^* - \lim_{\lambda\to\infty}$ in (2.2) and (2.4) holds uniformly for t in bounded intervals, $\|x^*\| \leq 1$ and that $T^{\times *}(t)x$ is a continuous X^{**} valued function of t for any $x \in X$.

Strangely enough we have not been able to prove these results directly. So we take a detour which is of its own interest. It is well known that $A^\times = A_0^* + C$ satisfies the resolvent estimates and therefore generates an 'integrated semigroup' $S^\times(t), t \geq 0$, on X^* which is locally Lipschitz in t with respect to the operator norm. See *Arendt* (1987), *Kellermann* (thesis), *Kellermann&Hieber* (1989). Actually it is possible to write down a variation of constants formula for $S^\times$, namely

$$\begin{aligned} S^\times(t) &= S_0^\times(t) + \int_0^t S_0^\times(t-\tau)d_\tau(CS^\times(\tau)) \\ &= S_0^\times(t) + \int_0^t S^\times(t-\tau)d_\tau(CS_0^\times(\tau)) \end{aligned} \tag{2.5}$$

with

$$S_0^\times(t) = \int_0^t T_0^*(r)dr$$

being the 'integrated semigroup' generated by A_0^*. The Stieltjes integrals in (2.5) hold in the operator norm. From the first formula in (2.5) we realize that $S^\times(t)x^*$ can be differentiated in the weak* sense yielding

$$\begin{aligned} T^\times(t)x^* &:= \frac{d^*}{dt}S^\times(t)x^* \\ &= T_0^*(t)x^* + \int_0^t T_0^*(t-\tau)d_\tau(CS^\times(\tau)x^*) \\ &= T_0^*(t)x^* + \int_0^t T^\times(t-\tau)d_\tau(CS_0^\times(\tau)x^*). \end{aligned} \tag{2.6}$$

The first integral in (2.6) is a weak* Stieltjes integral. The second equality in (2.6) will reveal that $X^{**} \ni T^{\times *}(t)x$ is a continuous function of t for $x \in X$. This will imply that the second integral in (2.6) makes sense as a weak* Stieltjes integral. As we will see in section 6 the second equality in (2.6) also shows that $(\lambda - A_0^*)^{-1}T^\times(t)$ is locally Lipschitz in t with respect to the operator norm because $(\lambda - A_0^*)^{-1}T_0^*(t)$ has this property. (2.6) will then imply (2.2).

The generation expansion (2.3), (2.4) is derived similarly using a generation expansion for $S^\times$. The following formula is particularly helpful in studying the dependence of $T^\times$ on C and T_0^*. Set $V_\infty(t) = CS^\times(t)$,

$V_0(t) = CS_0^\times(t)$ and consider the second equality in (2.6) and (2.5):

$$\begin{cases} T^\times(t)x^* = T_0^*(t)x^* + \int_0^t T_0^*(t-\tau)d_\tau(V_\infty(\tau)x^*) \\ V_\infty(t) = V_0(t) + \int_0^t V_0(t-\tau)d_\tau V_\infty(\tau) \\ \qquad = V_0(t) + \int_0^t V_\infty(t-\tau)d_\tau V_0(\tau) \end{cases} \tag{2.7}$$

In the next section a convolution calculus for locally Lipschitz operator kernels will be developed in which V_∞ plays the role of a resolvent kernel for V_0.

3. A CONVOLUTION CALCULUS FOR LOCALLY LIPSCHITZ CONTINUOUS OPERATOR KERNELS

3.1. *LIPSCHITZ KERNELS AND THEIR CONVOLUTION*

By a *kernel* (of operators) we mean a family $U(t), t \geq 0$, of linear bounded operators on a Banach space Y which satisfies

$$U(0) = 0 \tag{3.1}$$

and is locally Lipschitz in t (with respect to the operator norm), i.e. for any $t > 0$ there exists a $\Lambda_t > 0$ such that

$$\|U(r) - U(s)\| \leq \Lambda_t |r - s|, \quad 0 \leq r, s \leq t. \tag{3.2}$$

The kernels form a vector space in an obvious way. We define seminorms $\|\cdot\|_t$ by

$$\|U\|_t := \sup_{0 \leq r \neq s \leq t} \frac{\|U(r) - U(s)\|}{|r - s|}, \quad t > 0. \tag{3.3}$$

By (3.1), $U(0) = 0$, we have

$$\sup_{0 \leq r \leq t} \|U(r)\| \leq t\|U\|_t. \tag{3.4}$$

With these seminorms the kernels form a Fréchet space which becomes an algebra in the following way: For two kernels U, V we define the convolution $\star$ by

$$(U \star V)(t) = \int_0^t U(t-r)d_r V(r). \tag{3.5}$$

The integral in (3.5) is a Stieltjes integral in the operator norm, i.e. it is the limit of sums

$$\sum_{j=0}^{n} U(t-s_j)(V(r_{j+1})-V(r_j)), \quad s_j \in [r_j, r_{j+1}]$$

with $0 = r_0 < \ldots < r_{n+1} = t$, when the partition $r_0, \ldots, r_{n+1}$ gets finer. By reordering the sums one easily checks that

$$(U \star V)(t) = \int_0^t d_r U(r) V(t-r) \tag{3.6}$$

with the integral being the limit of sums

$$\sum_{j=0}^{n} (U(r_{j+1} - U(r_j)) V(t-s_j), \quad s_j \in [r_j, r_{j+1}],$$

$0 = r_0 < \ldots < r_{n+1} = t$. It is convenient to extend the kernels to $\mathbf{R}$ by setting

$$U(t) = 0, \quad t \leq 0. \tag{3.7}$$

Then they are locally Lipschitz on $\mathbf{R}$ and

$$(U \star V)(s) = \int_0^t U(s-r) d_r V(r), s \leq t. \tag{3.8}$$

$U \star V$ is a kernel again; actually we have the following inequalities in terms of the seminorms $\|\cdot\|_t$.

Lemma 3.1.

$$\|U \star V\|_t \leq \int_0^t \|U\|_{t-r} \, \|V\|_r dr \leq t\|U\|_t \|V\|_t .$$

Proof. Let $0 \leq r, s \leq t$. Then $(U \star V)(s) - (U \star V)(r)$ is approximated by sums

$$\sum_{j=0}^{n} (U(s-\sigma_{j+1}) - U(r-\sigma_{j+1}))(V(\sigma_{j+1}) - V(\sigma_j)) \tag{3.9}$$

with

$$0 = \sigma_0 < \cdots < \sigma_{n+1} = t .$$

The norm of the sum (3.9) can be estimated by

$$\sum_{j=0}^{n} \|U\|_{t-\sigma_{j+1}} \, |s-r| \, \|V\|_{\sigma_{j+1}} (\sigma_{j+1} - \sigma_j).$$

Taking the limit by refining the partitions we obtain

$$\|(U \star V)(s) - (U \star V)(r)\| \leq |s - r| \left(\int_0^t \|U\|_{t-\sigma} \|V\|_\sigma \, d\sigma \right).$$

This implies the first estimate. The second is trivial.

We can integrate $(U \star V)(t)$ and obtain a more familiar convolution.

Lemma 3.2. *$\int_0^t (U \star V)(r) dr = \int_0^t U(t-r)V(r) dr =: (U * V)(t)$. In other words,*

$$(U \star V)(t) = \frac{d}{dt}(U * V)(t)$$

with the differentiation holding in the operator norm.

Proof.

$$\int_0^t (U \star V)(r) dr = \int_0^t \int_0^t U(r-s) d_s V(s) \, dr = \int_0^t \left(\int_0^t U(r-s) dr \right) d_s V(s)$$

$$= \int_0^t \left(\int_0^{t-s} U(r) dr \right) d_s V(s) = -\int_0^t d_s \left(\int_0^{t-s} U(r) dr \right) V(s) = \int_0^t U(t-s) V(s) \, ds.$$

The second, fourth and fifth equality follow by approximating the integrals by sums and rearranging these, the first equality holds by definition, the third by standard integral calculus. Remember (3.7): $U(0) = V(0) = 0$. Noting that

$$\frac{d}{dt}(U * V) = U' * V = U * V',$$

provided the respective derivatives exist, we find that

$$U \star (V \star W) = \frac{d^2}{dt^2}(U * (V * W))$$

$$(U \star V) \star W = \frac{d^2}{dt^2}((U * V) * W).$$

As the associativity of $*$ is well-known and easily checked by standard integration theory, we have

Lemma 3.3. *$\star$ is associative, i.e. the Fréchet space of kernels is an algebra.*

In view of Lemma 3.1, the Fréchet space of kernels deserves the name Frechet algebra.

3.2 *RESOLVENT KERNELS*

The resolvent kernel V_∞ of a kernel V_0 is determined by the relation

$$V_\infty = V_0 + V_0 \star V_\infty = V_0 + V_\infty \star V_0 \,. \tag{3.10}$$

If it exists the resolvent kernel is unique by its algebraic properties. See *Gripenberg et al.* (1990), Section 9.3, Lemma 3.3.

Remark. Often the resolvent kernel of a kernel W_0 is defined by

$$W_\infty = W_0 - W_0 \star W_\infty = W_0 - W_\infty \star W_0. \tag{3.11}$$

See *Gripenberg et al.* (1990), Section 9.3. Note that (3.10) translates into (3.11) by setting $W_\infty = -V_\infty$, $W_0 = -V_0$. The concept of (3.11) seems to be more natural when 'frequency domain methods' are used whereas the concept of (3.10) is more convenient when exploiting order relations in case that Y is an ordered Banach space.

The standard construction of the resolvent kernel is the series of multiple convolutions:

$$V_\infty = \sum_{n=1}^{\infty} V^{\star n} \tag{3.12}$$

with

$$V^{\star 1} = V_0,\; V^{\star(n+1)} = V^{\star n} \star V_0. \tag{3.13}$$

The main point is showing the convergence of the series. (3.10) then follows from

Lemma 3.4. $V^{\star n} \star V_0 = V_0 \star V^{\star n}$

which is immediate by induction. From Lemma 3.1 we obtain by induction

Lemma 3.5. $\|V^{\star(n+1)}\|_t \leq \frac{t^n}{n!}\|V_0\|_t^{n+1}, \quad n \geq 1\,.$

So $\sum_{n=1}^{\infty} \|V^{\star n}\|_t \leq \|V_0\|_t \exp(t\|V_0\|_t)$ and the series (3.12) converges in the seminorms $\|\cdot\|_t$. By (3.4), $\sum_{n=1}^{\infty} V^{\star n}(t)$ converges in the operator norm uniformly for t in bounded intervals.

As a corollary we have the estimate

Lemma 3.6. $\|V_\infty\|_t \leq \|V_0\|_t \exp(t\|V_0\|_t)\,.$

The importance of resolvent kernels consists in solving convolution equations.

Lemma 3.7. (*Gripenberg et al.* (1990), Section 9.3, Lemma 3.5)

The convolution equation

$$U = U_0 + V_0 \star U$$

is uniquely solved by

$$U = U_0 + V_\infty \star U_0 \,,$$

whereas

$$W = W_0 + W \star V_0$$

is uniquely solved by

$$W = W_0 + W_0 \star V_\infty \,.$$

Before we estimate the solutions of convolution equations we make the following simple observation which follows from Lemma 3.6.

Lemma 3.8. $1 + \int_0^t \|V_\infty\|_r dr \leq \exp(t\|V_0\|_t)$.

Note that $\|V_0\|_t$ is a monotone non-decreasing function of t. The following is now easily derived from Lemma 3.7 and Lemma 3.1.

Lemma 3.9. *Let W solve $W = W_0 + W \star V_0$ or $W = W_0 + V_0 \star W$. Then*

$$\|W\|_t \leq \|W_0\|_t \exp(t\|V_0\|_t) \,.$$

We use this lemma for studying the dependence of the resolvent kernel V_∞ on V_0. Let

$$U_\infty = U_0 + U_0 \star U_\infty$$

$$V_\infty = V_0 + V_0 \star V_\infty \,.$$

Then

$$U_\infty - V_\infty = (U_0 - V_0) + (U_0 - V_0) \star U_\infty + V_0 \star (U_\infty - V_\infty).$$

By Lemma 3.9

$$\|U_\infty - V_\infty\|_t \leq \|U_0 - V_0 + (U_0 - V_0) \star U_\infty\|_t \exp(t\|V_0\|_t).$$

By Lemma 3.1

$$\|U_\infty - V_\infty\|_t \le \|U_0 - V_0\|_t\Big(1 + \int_0^t \|U_\infty\|_r dr\Big)\exp(t\|V_0\|_t).$$

By Lemma 3.8,

$$\|U_\infty - V_\infty\|_t \le \|U_0 - V_0\|_t \exp(t\|U_0\|_t)\exp(t\|V_0\|_t)\,.$$

So we have

Lemma 3.10. *Let U_∞, V_∞ be the resolvent kernels of U_0, V_0 respectively. Then*

$$\|U_\infty - V_\infty\|_t \le \|U_0 - V_0\|_t \exp(t[\|U_0\|_t + \|V_0\|_t])\,.$$

4. PERTURBATION OF LOCALLY LIPSCHITZ CONTINUOUS INTEGRATED SEMIGROUPS

It is well-known that an operator $A_0^\times$ on a Banach space Y generates an 'integrated semigroup' $S_0^\times(t)$, $t \ge 0$, on Y which is locally Lipschitz (with respect to the operator norm) iff $\lambda - A_0^\times$ can be continuously inverted for $\lambda > w$ and the resolvent estimates

$$\|(\lambda - A_0^\times)^{-n}\| \le \frac{M}{(\lambda - w)^n}, \quad \lambda > w, n \in \mathrm{N} \tag{4.1}$$

are satisfied. Actually

$$\|S_0^\times(t) - S_0^\times(r)\| \le M\, e^{wt}(t-r),\ t \ge r \ge 0\,. \tag{4.2}$$

Moreover we recall that by definition

$$S_0^\times(t)S_0^\times(r) = \int_0^t (S_0^\times(r+\tau) - S_0^\times(\tau))d\tau, \quad S_0^\times(0) = 0\,. \tag{4.3}$$

See *Arendt* (1987), *Kellermann* (thesis), *Kellermann&Hieber* (1989). The following relations hold between $A^\times$ and $S^\times$:

Lemma 4.1 *a) Let $x, y \in Y$. Then $x \in D(A_0^\times)$ and $A_0^\times x = y$ iff $\frac{d}{dt}S_0^\times(t)x = x + S_0^\times(t)y$ for all $t \ge 0$.*

b) $(\lambda - A_0^\times)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S_0^\times(t)dt = \int_0^\infty e^{-\lambda t} d_t S_0^\times(t)$.

c) For any $y \in Y$, $t \ge 0$, $\int_0^t S_0^\times(r)y dr \in D(A_0^\times)$ and $A_0^\times \int_0^t S_0^\times(r)ydr = S_0^\times(t)y - ty$.

See, e.g., *Thieme* (to appear).

If $C : \overline{D(A_0^\times)} \to Y$ is a bounded linear operator, the operator $A^\times = A_0^\times + C$ also satisfies the estimates (4.1) (with different w). See the proof of Theorem 1.1 in *Pazy* (1983), Section 3.1. So $A^\times$ generates a locally Lipschitz continuous integrated semigroup $S^\times$. Compare Proposition 3.3 in *Kellermann&Hieber* (1989). Actually it is possible to find $S^\times$ as the solution of the variation of constants formula

$$\begin{aligned} S^\times(t) &= S_0^\times(t) + \int_0^t S^\times(t-\tau)d_\tau(CS_0^\times(\tau)) \\ &= S_0^\times(t) + \int_0^t S_0^\times(t-\tau)d_\tau(CS^\times(\tau)). \end{aligned} \tag{4.4}$$

Taking Laplace transforms one realizes that

$$\lambda \int_0^\infty e^{-\lambda t} S^\times(t)dt = (\lambda - A^\times)^{-1}.$$

Applying C to (4.4) we realize that $CS^\times(t)$ coincides with the resolvent kernel V_∞ of V_0, $V_0(t) = CS_0^\times(t)$. Hence

$$S^\times(t) = S_0^\times(t) + \int_0^t S_0^\times(t-\tau)d_\tau V_\infty(\tau). \tag{4.5}$$

In other words,

$$S^\times = S_0^\times + S_0^\times \star V_\infty. \tag{4.6}$$

In turn, we can first construct V_∞ as the resolvent kernel of V_0 and define $S^\times$ by (4.6). If we multiply (4.6) by C and compare with (3.10) we find that $V_\infty = CS^\times$. Using the expansion (3.12), (3.13) we obtain the generation expansion

$$\begin{aligned} S^\times(t) &= \sum_{n=0}^\infty S_n^\times(t) \\ S_{n+1}^\times(t) &= \int_0^t S_n^\times(t-\tau)d_\tau(CS_0^\times(\tau)) = \int_0^t S_0^\times(t-\tau)d_\tau(CS_n^\times(\tau)). \end{aligned} \tag{4.7}$$

In fact the definition

$$S_n^\times = S_0^\times \star V^{\star n}, \quad n \geq 1,$$

yields $CS_{n+1}^\times = V^{\star(n+1)}$ — see (4.7) and Lemma 3.4 — and so

$$\begin{aligned} S_{n+1}^\times &= S_0^\times \star (CS_n^\times), \\ S_{n+1}^\times &= S_0^\times \star (V^{\star n} \star V_0) = (S_0^\times \star V^{\star n}) \star V_0 = S_n^\times \star (CS_0^\times). \end{aligned}$$

As a byproduct, we obtain the estimate

$$\sum_{n=0}^{\infty} \|S_{n+1}^{\times}\|_t < \infty, \quad \text{for any } t > 0\,. \tag{4.8}$$

5. PERTURBATION OF DUAL SEMIGROUPS

If $T_0^*(t)$ is the dual semigroup on X^* associated with a strongly continuous semigroup T_0 on X — the infinitesimal generator of which is A_0 —, then

$$S_0^{\times}(t)x^* = \int_0^t T_0^*(r)x^*\,dr \tag{5.1}$$

defines the locally Lipschitz continuous 'integrated semigroup' $S_0^{\times}(t)$ on X^* which is generated by A_0^*. Let $C\ :\ \overline{D(A_0^*)} \to X^*$ be a bounded linear operator. Then the perturbed operator $A^{\times} = A_0^* + C$ with $D(A^{\times}) = D(A_0^*)$ generates the integrated semigroup given by (4.4). From the second equation in (4.4) we realize that $S^{\times}(t)x^*$ can be differentiated in the weak* sense and that

$$\begin{aligned} T^{\times}(t)x^* \ :&= \frac{d^*}{dt}S^{\times}(t)x^* \\ &= T_0^*(t)x^* + \int_0^t T_0^*(t-\tau)d_\tau(CS^{\times}(\tau)x^*)\,. \end{aligned} \tag{5.2}$$

The integral on the right hand side has to be interpreted in the weak* sense. We note that $X^{**} \ni T^{\times *}(t)x$ is a continuous function of t for $x \in X$. Taking this into account we obtain from the first equation in (4.4) that

$$T^{\times}(t)x^* = T_0^*(t)x^* + \int_0^t T^{\times}(t-\tau)d_\tau(CS_0^{\times}(\tau)x^*), \tag{5.3}$$

where the integral on the right hand side has to be interpreted in a weak* sense:

$$\langle x, \int_0^t T^{\times}(t-\tau)d_\tau(CS_0^{\times}(\tau)x^*)\rangle$$

is the limit of the sums

$$\sum_{j=0}^{n}\langle \Big(CS_0^{\times}(\tau_{j+1}) - CS_0^{\times}(\tau_j)\Big)x^*, T^{\times *}(t-\sigma_j)x\rangle,$$

$0 = \tau_0 < \cdots < \tau_{n+1} = t, \sigma_j \in [t_j, t_{j+1}]$, when the partition $\tau_0, \cdots, \tau_{n+1}, n \in$ N, gets finer.

From (1.2) and the second equality in (4.4) we realize that $S^\times(t)x^*$ $\in D(A_0^*)$ and

$$A_0^* S^\times(t)x^* = -x^* + T^\times(t)x^* - CS^\times(t)x^* .$$

In other words

$$T^\times(t)x^* = x^* + A^\times S^\times(t)x^* . \tag{5.4}$$

This is property (1.2) for $T^\times, A^\times$. Using (4.3), (5.4) and Lemma 4.1c) we can verify that $T^\times(t)$ is a semigroup. From Lemma 4.1a) we obtain that

$$x^* \in D(A^\times), A^\times x^* = y^* \quad \text{iff} \quad T^\times(t)x^* - x^* = S^\times(t)y^* , t \geq 0 .$$

This is equivalent to (1.1) for $T^\times, A^\times$. Hence we have shown that the weakly* continuous semigroup $T^\times$ generated by $A^\times$ in the sense of (1.1), (1.2) is obtained by the formulas (2.6).

It is now easy to obtain a generation expansion for $T^\times$. Proceeding as before we can differentiate (4.7) in the weak* sense obtaining

$$\begin{aligned} T_{n+1}^\times(t)x^* \; &:= \frac{d^*}{dt} S_{n+1}^\times(t)x^* \\ &= \int_0^t T_0^*(t-\tau) d_\tau (CS_n^\times(\tau)) = \int_0^t T_n^\times(t-\tau) d_\tau (CS_0^\times(\tau)). \end{aligned} \tag{5.5}$$

It follows from (4.8) that

$$\sum_{n=0}^{\infty} T_n^\times(t)$$

converges in the operator norm uniformly on bounded intervals, hence the series in (4.7) can be differentiated in the weak* sense such that

$$T^\times(t)x^* = \frac{d^*}{dt} S^\times(t)x^* = \sum_{n=0}^{\infty} \frac{d^*}{dt} S_n^\times(t)x^* = \sum_{n=0}^{\infty} T_n^\times(t)x^* . \tag{5.6}$$

6. THE VARIATION OF CONSTANTS FORMULA (2.2)

In order to give a meaning to the integrals in (2.2) we prove

Lemma 6.1. *a) $(\lambda - A_0^*)^{-1} T_0^*(t)$ is locally Lipschitz in t with respect to the operator norm.*

b) The same holds for $(\lambda - A_0^*)^{-1} T^\times(t)$.

Proof: a)

$$(\lambda - A_0^*)^{-1}T_0^*(t) = T_0^*(t)(\lambda - A_0^*)^{-1}$$
$$= \int_0^t T_0^*(s)A_0^*(\lambda - A_0^*)^{-1}ds + (\lambda - A_0^*)^{-1}$$
$$= -\int_0^t T_0^*(s)ds + \int_0^t T_0^*(s)\lambda(\lambda - A_0^*)^{-1}ds + (\lambda - A_0^*)^{-1}.$$

b) By (5.2)

$$(\lambda - A_0^*)^{-1}T^\times(t) = (\lambda - A_0^*)^{-1}T_0^*(t) + \int_0^t (\lambda - A_0^*)^{-1}T_0^*(t-\tau)d_\tau(CS^\times(\tau)).$$

Part a) and Lemma 3.1 now imply the assertion.

In order to show the first equality in formula (2.2) we use formula (5.2) and prove that

$$w^* - \lim_{\lambda\to\infty} \int_0^t T_0^*(t-\tau)C\lambda(\lambda - A_0^*)^{-1}T^\times(\tau)x^*ds = \int_0^t T_0^*(t-\tau)d_\tau(CS^\times(\tau)x^*).$$

Note that the integrals on the left hand side can be approximated in the weak* sense by sums

$$\sum_j T_0^*(t-\tau_j)C\lambda(\lambda - A_0^*)^{-1}(S^\times(\tau_{j+1}) - S^\times(\tau_j))x^*$$

uniformly for large λ and uniformly for $\|x^*\| \le 1$, t in bounded intervals.

The integral on the right hand side can be approximated in the weak* sense by sums

$$\sum_j T_0^*(t-\tau_j)C(S^\times(\tau_{j+1}) - S^\times(\tau_j))x^*$$

uniformly for large λ and uniformly for $\|x^*\| \le 1$, t in bounded intervals.

So we only need to show that

$$\lambda(\lambda - A_0^*)^{-1}S^\times(r) \to S^\times(r),\ \lambda \to \infty$$

uniformly for r in bounded intervals. But

$$\lambda(\lambda - A_0^*)^{-1}S^\times(r) - S^\times(r) = (\lambda - A_0^*)^{-1}A_0^*S^\times(r)$$
$$= (\lambda - A_0^*)^{-1}(T^\times(r) - I - CS^\times(r))$$

— see (5.4) — and

$$\|(\lambda - A_0^*)^{-1}\| \to 0 \quad \text{for } \lambda \to \infty.$$

The second equality in (2.2) is shown similarly using (5.3). Note that $T^{\times *}(t)x, x \in X$, is a continuous X^{**} valued function of $t \geq 0$. (2.4) is derived from (5.5) in the same way. Note from (4.7) and (1.2) that $S_{n+1}^{\times}(t)x^* \in D(A_0^*)$ and

$$A_0^* S_{n+1}^{\times}(t)x^* = T_{n+1}^{\times}(t)x^* - x^* - CS_n^{\times}(t)x^* .$$

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